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d-Koszul algebras, 2-d-determined algebras and 2-d-Koszul algebras

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ABSTRACT

The relationship between an algebra and its associated monomial algebra is investigated when at least one of the algebras is d-Koszul. It is shown that an algebra which has a reduced Gröbnerbasis that is composed of homogeneous elements of degree d is d-Koszul if and only if its associated monomial algebra is d-Koszul. The class of 2-d-determined algebras and the class 2-d-Koszul algebras are introduced. In particular, it is shown that 2-d-determined monomial algebras are 2-d-Koszul algebras and the structure of the ideal of relations of such an algebra is completely determined.

1. Introduction

This paper focuses on the study of classes of graded algebras whose graded projective resolutions of the semisimple part have special properties. We also investigate the relationship between the algebra having projective resolutions with certain special properties and the structure of the Ext-algebra of the semisimple part of such an algebra. In the past, some of the strongest results have been obtained for Koszul algebras, a special class of graded algebras that has occurred in many diverse settings. Generalizations of Koszul algebras, for example, d-Koszul algebras [2,12], have recently been studied. In this paper, we continue the investigation of d-Koszul algebras and begin a study of a new classes of algebras which we call 2-d-determined algebras and 2-d-Koszul algebras. We begin by summarizing the major results of the paper. Precise definitions for many of the terms used can be found later in this section and the next section.

In the summary below, we let $\Lambda = K\Gamma/I$, where K is a field, Γ a finite quiver, $K\Gamma$ the path algebra, and I an ideal generated by length homogeneous elements. Let J be the ideal in $K\Gamma$ generated by the arrows of Γ and assume that $I \subset J^2$. The length grading of $K\Gamma$ induces a positive $\mathbb{Z}$-grading of $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$, where Λ_0 is the K-space spanned by the vertices of Γ. In particular, $\Lambda_0 \cong \Lambda/(J/I)$ and $J/I \cong \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ is a graded Jacobson radical of Λ.

After the summary of results, this section ends with the introduction of notation, background, and a brief overview of the theory of Gröbnerbases for path algebras. In Section 2 we recall constructions of projective resolutions found in [1,15], which we call the 'AGS resolution', and also review the general approach to the structure of projective resolutions found in [17]. Given $\Lambda = K\Gamma/I$, using the theory of Gröbner bases, we associate a monomial algebra, Λ_{mon}, to Λ, where by 'monomial algebra', we mean a quotient of a path algebra by an ideal that can be generated by a set of paths. In this case, $\Lambda_{mon} = K\Gamma/I_{mon}$, where I_{mon} is the ideal generated by the 'tips' or 'leading terms' of I. One of the main objectives of the paper is the study of the interrelationship of Λ and Λ_{mon}. In Section 3, we turn our attention to d-Koszul algebras, which were introduced by Berger [2]. Let $\mathbb{N}$ denote the natural numbers $\{0, 1, 2, \dots\}$ and let $d \in \mathbb{N}$ with $d \geq 2$. Consider the function $\delta\colon \mathbb{N} \to \mathbb{N}$ defined by

$$\delta(n) = \begin{cases} \frac{n}{2}d & \text{if } n \text{ is even} \\ \frac{n-1}{2}d + 1 & \text{if } n \text{ is odd.} \end{cases}$$

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We say that $\Lambda = K\Gamma/I$ is a *d-koszul algebra* if the nth-projective module in a minimal graded projective Λ-resolution of Λ_0 can be generated in degree $\delta(n)$. More generally, if $F: \mathbb{N} \to \mathbb{N}$, we say that Λ is *F-determined*, (respectively *weakly F-determined*) in the case where the nth-projective module in a minimal graded projective Λ-resolution of Λ_0 can be generated in degree $F(n)$, (resp. $\leq F(n)$), for all $n \in \mathbb{N}$. The notion of F-determined algebras was introduced in [11] and also investigated in [13]. Proposition 7 shows that, in particular, if Λ_{mon} is weakly F-determined, then so is Λ. We use this result to show that if Λ_{mon} is a d-Koszul algebra then so is Λ in Corollary 8. Theorem 10 gives a partial converse, showing that if $\Lambda = K\Gamma/I$ and I has a reduced Gröbner basis concentrated in degree d, then Λ_{mon} is a d-Koszul algebra. Theorem 12 summarizes the main results of the section.

In Section 4, we introduce the class of 2-d-determined algebras. We say that Λ is *2-d-determined* if I can be generated by homogeneous elements of degrees 2 and d, and Λ is weakly δ-determined. We say a 2-d-determined algebra is *2-d-Koszul* if the Ext-algebra, $\bigoplus_{n\geq 0} \mathrm{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)$, can be finitely generated. In this section, we mainly consider the case where Λ is a monomial algebra. Theorem 14 proves that if $\Lambda = K\Gamma/I$ and I is generated by paths of lengths 2 and d, then Λ is a 2-d-determined algebra if and only if $K\Gamma/\langle \mathcal{G}_d\rangle$ is a d-Koszul algebra, where $\mathcal{G}_d$ denotes the set of paths of length d in a minimal generating set of I. In Theorem 16, we show that a monomial algebra with generators in degrees 2 and d is 2-d-determined if and only if the Ext-algebra, $\bigoplus_{n\geq 0} \mathrm{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)$, can be generated in degrees 0, 1, and 2. Algebras, whose Ext-algebra can be generated in degrees 0, 1, and 2 have been called *K2 algebras* by Cassidy and Shelton [6].

In the final section, Section 5, we study 2-d-determined algebras in general. Proposition 17 shows that if Λ_{mon} is 2-d-Koszul then Λ is 2-d-determined. The next result is the main result of the section.

Theorem 18. *Let $\Lambda = K\Gamma/I$, where I is a homogeneous ideal in $K\Gamma$, and let $>$ be an admissible order on $\mathcal{B}$. Suppose that the reduced Gröbner basis $\mathcal{G}$ of I with respect to $>$ satisfies $\mathcal{G} = \mathcal{G}_2 \cup \mathcal{G}_d$ where $\mathcal{G}_2$ consists of homogeneous elements of degree 2 and $\mathcal{G}_d$ consists of homogeneous elements of degree d, where $d \geq 3$. Then Λ is 2-d-determined if $K\Gamma/\langle \mathrm{tip}(\mathcal{G}_d)\rangle$ is a d-Koszul algebra.*

Section 5 ends with some open questions.

We end this section with some definitions and notations that will be used throughout the remainder of the paper. We always let Γ denote a finite quiver and $K\Gamma$ its path algebra over a fixed field K. The K-algebra $K\Gamma$ is naturally a positively $\mathbb{Z}$-graded algebra, where, if n is a nonnegative integer, then $(K\Gamma)_n$ denotes the homogeneous component of $K\Gamma$ which is the vector space with basis the set of paths of length n. We denote the length of a path p by $\ell(p)$ and let Γ_n denote the set of directed paths of length n in Γ; in particular, Γ_0 is the set of vertices of Γ and Γ_1 is the set of arrows in Γ. We call this the *length grading* of $K\Gamma$ and say that an element of $K\Gamma$ is *homogeneous* if all the paths occurring in the element have the same length. In particular, if $f \in (K\Gamma)_n$, then we say that f is *homogeneous of (length) degree n* and write $\ell(f) = n$. If I is an ideal in $K\Gamma$, we say that I is a *homogeneous ideal* if I can be generated by homogeneous elements. Clearly, if I is a homogeneous ideal in $K\Gamma$, then $K\Gamma/I$ has a grading induced from the length grading of $K\Gamma$, and we call this the *length grading on $K\Gamma/I$ induced by the length grading on $K\Gamma$*, or simply, the *induced length grading on $K\Gamma/I$*.

If an ideal I can be generated by a set of paths in Γ, then we say that I is a *monomial ideal* and that $\Lambda = K\Gamma/I$ is a *monomial algebra*. Since every monomial ideal is a homogeneous ideal, every monomial algebra has an induced length grading.

As mentioned earlier, we denote by J, the ideal of $K\Gamma$ generated by the arrows of Γ. By 'module', we mean 'left module' unless otherwise stated. If $\Lambda = K\Gamma/I$, where I is an ideal contained in J and we denote by Λ_0, the semisimple Λ-module $\Lambda_0 = (\Lambda/I)/(J/I)$. Suppose further that I is a homogeneous ideal and $\Lambda = K\Gamma/I$ is given the induced length grading. Note that J/I is the graded Jacobson radical of Λ. The Λ-module Λ_0 will also be viewed as a graded Λ-module whose support is concentrated in degree 0. If $S_1, \ldots, S_n$ is a full set of nonisomorphic simple Λ-modules, then $\Lambda_0 \cong \oplus_{i=1}^n S_i$, as an (ungraded) Λ-module. We also note that, in the category of graded Λ-modules, Λ_0 has a *minimal graded projective resolution*

$$\cdots \to P^2 \to P^1 \to P^0 \to \Lambda_0 \to 0,$$

in the sense that each P^n is a graded projective Λ-module, each map $P^n \to P^{n-1}$ is a degree 0 homomorphism, and, for each $n \geq 1$, the image of P^n in P^{n-1} is contained in $(J/I)P^{n-1}$.

If $\Lambda = K\Gamma/I$, for some homogeneous ideal I in $K\Gamma$, and $v \in \Gamma_0$, then we view Λv as an indecomposable graded Λ-module generated in degree 0 by v. If $M = \oplus_{i\in\mathbb{Z}} M_i$ is a graded Λ-module, then we let $M[n]$ denote the nth-shift of M; that is, $M[n] = \oplus_{i\in\mathbb{Z}} N_i$, where $N_i = M_{i+n}$. It is well known, for example, see [7], that every graded indecomposable projective Λ-modules is isomorphic to $\Lambda v[n]$, for some unique $n \in \mathbb{Z}$ and $v \in \Gamma_0$ and that every finitely generated graded projective Λ-module can be written as a direct sum of projective modules of the form $\Lambda v[n]$.

Given a set X in $K\Gamma$, we denote by $\langle X\rangle$, the two sided ideal in $K\Gamma$ generated by X. We will freely use the terminology and results about Gröbner bases for path algebras found in [8]. For the reader's benefit, we recall some of the definitions. We say a nonzero element $x \in K\Gamma$ is *uniform* if there exist vertices $v, w \in \Gamma_0$ such that $vx = x = xw$. Note that any nonzero element of $K\Gamma$ is a sum of uniform elements and that any ideal in $K\Gamma$ can be generated by uniform elements.

Let $\mathcal{B} = \cup_{n\geq 0}\Gamma_n$ be the set of paths in Γ. We say that a well ordering $>$ on $\mathcal{B}$ is an *admissible order* if the following conditions hold for all $p, q, r, s \in \mathcal{B}$.

(1) If $p > q$, then $rp > rq$, if both are nonzero.
(2) If $p > q$, then $pr > qr$, if both are nonzero.
(3) If $p = qrs$, then $p \geq r$.

If $x \in K\Gamma$, then $x = \sum_{p\in\mathcal{B}} \alpha_p p$, where $\alpha_p \in K$ and almost every $\alpha_p = 0$. If $x \neq 0$, the *tip of* x, denoted $\mathrm{tip}(x)$, is the path $p \in \mathcal{B}$ such that $\alpha_p \neq 0$ and $p \geq q$, for all q such that $\alpha_q \neq 0$. If $X \subset K\Gamma$, then $\mathrm{tip}(X) = \{\mathrm{tip}(x) \mid x \in X \setminus \{0\}\}$. We say a path p *occurs in* $\sum_{q\in\mathcal{B}} \alpha_q q \in K\Gamma$ if $\alpha_p \neq 0$.

Fix an admissible order $>$ on $\mathcal{B}$ and let I be an ideal in $K\Gamma$. We say a set $\mathcal{G}$ of nonzero uniform elements in I is a *Gröbner basis of I (with respect to $>$)* if $\langle \mathrm{tip}(\mathcal{G})\rangle = \langle \mathrm{tip}(I)\rangle$. We say a Gröbner basis $\mathcal{G}$ of I is the *reduced Gröbner basis* if, for every $g \in \mathcal{G}$, the coefficient of $\mathrm{tip}(g)$ is 1 and, if p is a path occurring in g and p contains a subpath t, where $t = \mathrm{tip}(g')$, for some $g' \in \mathcal{G}$, then $g = g'$. We note that given $>$ and an ideal I in $K\Gamma$, the reduced Gröbner basis of I exists and is unique. Using the Buchberger algorithm as generalized for path algebras, one can see that if I is a homogeneous ideal in $K\Gamma$, the reduced Gröbner basis of I consists of homogeneous uniform elements.

Now suppose that $\mathcal{G}$ is the reduced Gröbner basis of an ideal I with respect to $>$. Let I_{mon}, the *associated monomial ideal to* I, be the ideal in $K\Gamma$ generated by the tips of $\mathcal{G}$. If $\Lambda = K\Gamma/I$, we let $\Lambda_{mon} = K\Gamma/I_{mon}$. We call Λ_{mon} the *associated monomial algebra of* Λ with respect to $>$. We note that if I is a monomial ideal, then $I = I_{mon}$, and this is independent of the choice of $>$; whereas, if I is not a monomial ideal, then I_{mon} usually depends on the choice of admissible order.

2. The AGS resolution

Although the proofs of the results in this section appear in other papers, they are not stated or combined together in a fashion that we need throughout the remainder of the paper. Hence we have included this survey for the readers' benefit.

In both [1] and [15], methods for constructing a projective Λ-resolution of Λ_0 are given and we will call such a constructed resolution the *AGS resolution*. The reader may check that these constructed resolutions of Λ_0 are, in fact, the same; although in [15], resolutions of a larger class of modules, that includes Λ_0, are given. Both methods employ an admissible order on $\mathcal{B}$ and a Gröbner basis of I (with respect to the chosen admissible order). Furthermore, the reader may check that, if I can be generated by length homogeneous elements, then the AGS resolution is, in fact, a resolution in the category of graded Λ-modules, see [14]. In general, the AGS resolution is not minimal, but, if the Gröbner basis is finite, the projective modules occurring in the resolution, viewed as graded Λ-modules, can be written as finite direct sums of projective Λ-modules of the form $\Lambda v[n]$, where v is a vertex in Γ, and $n \in \mathbb{Z}$. Note that, if I has a Gröbner basis that consists only of paths (of length at least 2), then the AGS resolution is minimal.

In a path algebra $K\Gamma$, if $x \in K\Gamma$ is a nonzero element such that $vx = x$, where $v \in \Gamma_0$, then we let $o(x) = v$. Similarly, if $xv = x$, where $v \in \Gamma_0$, then we let $t(x) = v$.

Let $>$ be an admissible order, I a homogeneous ideal in $K\Gamma$, $\Lambda = K\Gamma/I$, and $\mathcal{G}$ be the reduced Gröbner basis for I. Suppose that $\mathcal{G} = \{g_i\}_{i\in\mathcal{I}}$, for some index set $\mathcal{I}$, i.e. it is the Gröbner basis, which we use to describe the second projective on the minimal projective resolution on the monomial algebra, accordingly with the notation used in [14]. If v is a vertex, by abuse of notation, we will let Λv also denote the graded projective Λ-module generated by v with v in degree 0. Suppose that

$$\cdots \to Q^2 \to Q^1 \to Q^0 \to \Lambda_0 \to 0$$

is the AGS (graded) resolution of Λ_0. Then $Q^0 = \oplus_{v\in\Gamma_0}\Lambda v$, $Q^1 = \oplus_{a\in\Gamma_1}\Lambda o(a)[-1]$, and $Q^2 = \oplus_{i\in\mathcal{I}}\Lambda o(g_i)[-\ell(g_i)]$.

We briefly describe the structure of Q^3, leaving details to be found in [15]. For this, we need a few more definitions. If $p, q \in \mathcal{B}$, we say p *overlaps* q if there are paths r and s such that $pr = sq$ and $\ell(s) < \ell(p)$, and that the overlap is *proper* if $\ell(r) \geq 1$ and $\ell(s) \geq 1$. We say q is a *subpath* of p if $p = rqs$ for some paths r and s, and that q is a *proper subpath* of p, if $p = rqs$, for some paths r and s, with $\ell(r) \geq 1$ and $\ell(s) \geq 1$. As remarked earlier, since $\mathcal{G}$ is a reduced Gröbner basis, if $g_i = \sum_{j=1}^{m} \alpha_j q_j$, where each α_j is a nonzero element of K and the q_j's are distinct paths, then if $s \neq i$, $\mathrm{tip}(g_s)$ is not a subpath of q_j, for $j = 1, \ldots, m$.

If ρ is a set of paths of length at least 2, and $t, q \in \mathcal{B}$, then we say a path p is *the maximal overlap of t with q with respect to ρ* if the following two conditions hold.

(1) If t overlaps q such that there exist paths $s, s' \in \mathcal{B}$ with $\ell(s') \geq 1$ with $p = s'q = ts$.
(2) For all $t' \in \rho$, t' is not a proper subpath of p.

In this case, we say that p *maximally overlaps* q with respect to ρ.

We will be interested in maximal overlaps of elements of $\mathrm{tip}(\mathcal{G})$ with various paths with respect $\mathrm{tip}(\mathcal{G})$. In particular, let

$$T^3 = \{p \in \mathcal{B} \mid p \text{ is the maximal overlap of } t' \text{ with } t \text{ with respect to } \mathrm{tip}(\mathcal{G}),\ \text{where } t, t' \in \mathrm{tip}(\mathcal{G})\}.$$

Note that if $\mathcal{G}$ is a finite set, then T^3 is also a finite set. In general, we have the following.

Proposition 1 ([15]). *Let $\Lambda = K\Gamma/I$ where I is a homogeneous ideal in $K\Gamma$ and suppose that $>$ is an admissible order on $\mathcal{B}$. Let*

$$\cdots \to Q^2 \to Q^1 \to Q^0 \to \Lambda_0 \to 0$$

be the AGS (graded) resolution of Λ_0. Then, as graded Λ-modules,

$$Q^3 = \oplus_{t\in T^3}\Lambda o(t)[-\ell(t)],$$

where $T^3 = \{p \in \mathcal{B} \mid p$ is the maximal overlap of t' and t, for some $t, t' \in \mathrm{tip}(\mathcal{G})\}$.

The AGS resolution is a special case of projective Λ-resolutions of modules which are studied in [14]. We recall some definitions and results from that paper, since the perspective and notation developed there will be used in some of the proofs that follow. For ease of notation, we will sometimes denote $K\Gamma$ by R.

Let M be a $K\Gamma$-module and m be a nonzero element of M. We say that m is *left uniform* if there exist u in Γ_0 such that $m = um$. In this case, we let $o(m) = u$. Note that if Γ has a single vertex then every nonzero of M is left uniform.

Suppose that M is a finitely generated Λ-module. Then, as shown in [14], there exist t_n and u_n in $\{0, 1, 2, \dots\} \cup \infty$ with $u_0 = 0$, $\{f_i^n\}_{i\in T_n=[1,\dots,t_n]}$, and $\{f_i'^n\}_{i\in U_n=[1,\dots,u_n]}$ such that

(i) Each f_i^0 is a left uniform element of R, for all $i \in T_0$.
(ii) Each f_i^n is in $\oplus_{j\in T_{n-1}} Rf_j^{n-1}$ and is a left uniform element, for all $i \in T_n$ and all $n \geq 1$.
(iii) Each $f_i'^n$ is in $\oplus_{j\in T_{n-1}} If_j^{n-1}$ and is a left uniform element for all $i \in U_n$ and all $n \geq 1$.
(iv) For each $n \geq 2$,

$$(\oplus_{i\in T_{n-1}} Rf_i^{n-1}) \cap (\oplus_{i\in T_{n-2}} If_i^{n-2}) = (\oplus_{i\in T_n} Rf_i^n) \oplus (\oplus_{i\in U_n} Rf_i'^n).$$

An explicit description of the tip set of T_n for the AGS resolution of Λ_0, is given in Proposition 5 below. The next result explains how the sets $\{f_i^n\}_{i\in T_n}$ and $\{f_i'^n\}_{i\in U_n}$ give rise to a projective Λ-resolution of M. We have the following isomorphisms:

$$\oplus_{i=1}^m Rf_i / \oplus_{i=1}^m If_i \cong \oplus_{i=1}^m (Rf_i/If_i) \cong \oplus_{i=1}^m \Lambda o(f_i).$$

Theorem 2 ([14]). *Let M be a finitely generated Λ-module and suppose that, for $n \geq 0$, t_n and u_n are in $\{0, 1, 2, \dots\} \cup \infty$, $\{f_i^n\}_{i\in T_n=[1,\dots,t_n]}$, and $\{f_i'^n\}_{i\in U_n=[1,\dots,u_n]}$ are chosen satisfying* (i)-(iv) *above. Let*

$$L^n = \bigoplus_{i\in T_n} \Lambda o(f_i^n).$$

Then there exist maps $e^n : L^n \to L^{n-1}$ and a surjection $L^0 \to M$ such that

$$\cdots \xrightarrow{e^{n+1}} L^n \xrightarrow{e^n} L^{n-1} \xrightarrow{e^{n-1}} \cdots \xrightarrow{e^1} L^0 \to M \to 0$$

is a projective Λ-resolution of M.

Although we do not use explicit descriptions of the maps e^n in this paper, we note that such descriptions can be found in [14]. The AGS resolution is obtained by constructing particular f_i^n's which satisfy (i)–(iv). By Proposition 1, we see that, for this choice of the f_i^n's, $\{\mathrm{tip}(f_i^3)\}_{i\in T_3}$ is precisely the set of maximal overlaps T^3 defined earlier. From this observation, we have the following useful result.

Proposition 3. *Let $>$ be an admissible order, I a homogeneous ideal in $K\Gamma$, $\Lambda = K\Gamma/I$, and $\mathcal{G}$ be the reduced Gröbner basis for I. Suppose that*

$$\cdots \to P^2 \to P^1 \to P^0 \to \Lambda_0 \to 0$$

is a minimal graded projective Λ-resolution of Λ_0. Then P^3 is isomorphic to $\bigoplus_{t\in(T^*)^3} \Lambda o(t)[-l(t)]$, for some subset $(T^*)^3$ of the set of maximal overlaps of $\mathcal{G}$ with respect to $\mathcal{G}$.

Proof. Let $\cdots \to Q^2 \to Q^1 \to Q^0 \to \Lambda_0 \to 0$ be the (graded) AGS resolution of Λ_0. By Proposition 1, $Q^3 = \bigoplus_{t\in T^3} \Lambda o(t)[-l(t)]$, where T^3 is the set of maximal overlaps of $\mathrm{tip}(\mathcal{G})$ with respect to $\mathrm{tip}(\mathcal{G})$. The result now follows from [14, Theorem 2.4] after by applying the proof of [14, Theorem 2.3]. □

In a similar fashion, the following more general result is a consequence of the proof of Theorem 2.3 and Theorem 2.4 in [14].

Proposition 4. *Let $>$ be an admissible order, I a homogeneous ideal in $K\Gamma$, and $\Lambda = K\Gamma/I$. Suppose that*

$$\cdots \to P^2 \to P^1 \to P^0 \to \Lambda_0 \to 0$$

is a minimal graded projective Λ-resolution of Λ_0 and that

$$\cdots \to Q^2 \to Q^1 \to Q^0 \to \Lambda_0 \to 0$$

is the (graded) AGS resolution of Λ_0. If $Q^n \cong \bigoplus_{f_j^n\in T_n} \Lambda o(f_j^n)[-\ell(f_j^n)]$, then P^n is isomorphic to $\bigoplus_{f_j^n\in(T_n)^*} \Lambda o(f_j^n)[-\ell(f_j^n)]$, for some subset $(T_n)^*$ of T_n.

We introduce some notation that will be needed later in the paper. Let ρ be a set of paths of length at least 2 such that no path in ρ is a subpath of any other path in ρ. We define the sets $\mathrm{AP}(n)$ of *admissible paths of order n, with respect to ρ*. First let $\mathrm{AP}(0) = \Gamma_0$, $\mathrm{AP}(1) = \Gamma_1$, and $\mathrm{AP}(2) = \rho$. Next, we let $\mathrm{AP}(3)$ be the set of all maximal overlaps of elements of ρ with elements of ρ, with respect to ρ. Assume $\mathrm{AP}(n-2)$ and $\mathrm{AP}(n-1)$ have been defined. Then define $\mathrm{AP}(n)$ to be the set of paths a_n in Γ which satisfy the following conditions.

(A1) There are paths $a_{n-1} \in \mathrm{AP}(n-1)$ and $r \in \mathcal{B}$ such that $\ell(r) \geq 1$ and $a_n = ra_{n-1}$.
(A2) If $a_{n-1} = sa_{n-2}$ with $a_{n-2} \in \mathrm{AP}(n-2)$, then $rs = a_2s'$ for some $s' \in \mathcal{B}$, some $a_2 \in \rho$, and we have that $\ell(r) < \ell(a_2)$.
(A3) a_2s' does not contain any element of ρ as a proper subpath.

The reader may check that for $n \geq 2$, given $a_n \in \mathrm{AP}(n)$, then there exist unique paths $r \in \mathcal{B}$ and $a_{n-1} \in \mathrm{AP}(n-1)$ such that $a_n = ra_{n-1}$. Again, it is not hard to show that, in the above notation the overlap of a_2 with s is maximal with respect to ρ. For $n \geq 2$, we associate to a path $a_n \in \mathrm{AP}(n)$, its *admissible sequence (with respect to ρ)*, which is defined to be $(p_{n-1}, p_{n-2}, \ldots, p_1)$, where

(1) Each $p_i \in \rho = \mathrm{AP}(2)$,
(2) $a_n = p_{n-1}s_{n-1}a_{n-2}$, for some $s_{n-1} \in \mathcal{B}$ and $a_{n-2} \in \mathrm{AP}(n-2)$,
(3) $a_n = ra_{n-1}$, for some $r \in \mathcal{B}$ and $a_{n-1} \in \mathrm{AP}(n-1)$, and, if $n \geq 3$, then
(4) $(p_{n-2}, \ldots, p_1)$ is the admissible sequence for a_{n-1} with respect to ρ.

Proposition 5 (*[15,9]*). *Let $\Lambda = K\Gamma/I$, for some homogeneous ideal of $K\Gamma$. Fix some admissible order $>$ on $\mathcal{B}$, and let $\mathcal{G}$ denote the reduced Gröbner basis for I with respect to $>$. For $n \geq 0$, let $\{f_i^n\}_{i\in T_n}$ denote the elements defined in [15] in the construction of the AGS Λ-resolution of Λ_0. Let $\mathrm{AP}(n)$ be the set of admissible paths of order n with respect to $\mathrm{tip}(\mathcal{G})$. Then, for $n \geq 0$,*

$$\{\mathrm{tip}(f_i^n)\}_{i\in T_n} = AP(n).$$

Finally, the following result relates the AGS Λ-resolution of Λ_0 to a minimal projective Λ_{mon}-resolutions of $(\Lambda_{mon})_0$. The proof follows from [9].

Proposition 6. *Let $\Lambda = K\Gamma/I$, where I is a homogeneous ideal in $K\Gamma$ generated by elements of homogeneous length at least 2. Fix an admissible order $>$ on $\mathcal{B}$ and let $\mathrm{AP}(n)$ denote the admissible paths of order n with respect to $\mathrm{tip}(\mathcal{G})$, where $\mathcal{G}$ is the reduced Gröbner basis of I with respect to $>$. Let $\cdots \to L^2 \to L^1 \to L^0 \to (\Lambda_{mon})_0 \to 0$ be a minimal projective Λ_{mon}-resolution of $(\Lambda_{mon})_0$ and $\cdots \to Q^2 \to Q^1 \to Q^0 \to \Lambda_0 \to 0$ be the AGS Λ-resolution of Λ_0. Then*

$$L^n \cong \bigoplus_{p\in\mathrm{AP}(n)} \Lambda_{mon}o(p).$$

and

$$Q^n \cong \bigoplus_{p\in\mathrm{AP}(n)} \Lambda o(p).$$

3. *d*-Koszul algebras

Fix the following notation for the remainder of this section. We let K denote a field, Γ a quiver, $K\Gamma$ the path algebra, I a homogeneous ideal in $K\Gamma$ contained in $J^2 = \langle \text{arrows in } \Gamma\rangle^2$, $\Lambda = K\Gamma/I$, which is given the induced length grading, $>$ an admissible order, and $\mathcal{G} = \{g_i^2\}_{i\in\mathcal{I}}$ is the reduced Gröbner basis for I with respect to $>$, where $\mathcal{I}$ is an index set. Let $\cdots \to P_\Lambda^2 \to P_\Lambda^1 \to P_\Lambda^0 \to \Lambda_0 \to 0$ be a minimal graded projective Λ-resolution. Recall that if $F: \mathbb{N} \to \mathbb{N}$ and, for each $n \geq 0$, P^n can be generated in degree $F(n)$, we say that Λ is *F-determined*. If, for each $n \geq 0$, P^n can be generated in degrees $\leq F(n)$, we say that Λ is *weakly F-determined*. If $\delta: \mathbb{N} \to \mathbb{N}$ is defined by

$$\delta(n) = \begin{cases} \frac{n}{2}d & \text{if } n \text{ is even} \\ \frac{n-1}{2}d + 1 & \text{if } n \text{ is odd,} \end{cases}$$

and Λ is δ-determined, we say that Λ is *d-Koszul*. Note if $d = 2$, then we see that 2-Koszul is the same as Koszul. We use the term "d-Koszul" and note that this usage is consistent with our definition of δ-Koszul, which states that an algebra is called δ-Koszul when it is δ-determined and its Ext algebra is finitely generated. If an algebra is d-Koszul, for $d > 2$, then its Ext-algebra is finitely generated (in degrees 0, 1, and 2) [12]; whereas if the algebra is Koszul, then its Ext-algebra generated in degrees 0 and 1, [16]. Also note that, since $\delta(2) = d$, the ideal I is homogeneous and can be generated by homogeneous elements of degree d.

In [1], there is a number of results of the following form: if Λ_{mon} has some property, then so does Λ. The two next results are of this nature.

Proposition 7. *Let $\Lambda = K\Gamma/I$ be as above. Suppose that $F: \mathbb{N} \to \mathbb{N}$ is a set function such the Λ_{mon} is weakly F-determined. Then Λ is weakly F-determined. Furthermore, if F is not strictly increasing, then Λ_0 has finite projective dimension both as a Λ-module, and as a Λ_{mon}-module. In particular, if $F(s+1) > F(s)$, for $0 \leq s \leq m$, and $F(m+1) \leq F(m)$, then the projective dimensions of $\Lambda_0 \leq m$.*

Moreover, if Λ_{mon} is F-determined, then Λ is F-determined and the AGS Λ-resolution of Λ_0 is minimal.

Proof. First assume that Λ_{mon} is either weakly F-determined or F-determined. Let $\cdots \to L^2 \to L^1 \to L^0 \to (\Lambda_{mon})_0 \to 0$ be the graded AGS Λ_{mon}-resolution of $(\Lambda_{mon})_0$, which is a minimal projective Λ_{mon}-resolution of $(\Lambda_{mon})_0$. Let $\cdots \to P^2 \to P^1 \to P^0 \to \Lambda_0 \to 0$ be a minimal graded Λ-resolution of Λ_0 and $\cdots \to Q^2 \to Q^1 \to Q^0 \to \Lambda_0 \to 0$ be the graded AGS Λ-resolution of Λ_0. Let $\{f_i^n\}_{i\in T_n}$ be the elements constructed in the AGS Λ-resolution of Λ_0. Then, by Proposition 4, for each $n \geq 0$, there are subsets T_n^* of T_n such that $P^n \cong \oplus_{j\in T_n^*}\Lambda o(f_i^n)[-\ell(f_i^n)]$. By Proposition 6, for each $n \geq 0$, $L^n \cong \oplus_{i\in T_n}\Lambda_{mon}o(\text{tip}(f_i^n))[-\ell(\text{tip}(f_i^n))]$. Since Λ_{mon} is either weakly F-determined or F-determined, we see that, for each $n \geq 0$, either $\ell(\text{tip}(f_i^n)) \leq F(n)$ or $\ell(\text{tip}(f_i^n)) = F(n)$, for all $i \in T_n$. Since $\ell(f_i^n) = \ell(\text{tip}(f_i^n))$, which is the degree of f_i^n, for every i and n, we see that Λ is weakly F-determined if Λ_{mon} or F-determined if Λ_{mon} is.

Now assume further that $F(s+1) > F(s)$, for $0 \leq s \leq m$ and $F(m+1) \leq F(m)$. Since $\cdots \to L^2 \to L^1 \to L^0 \to (\Lambda_{mon})_0 \to 0$ is minimal graded AGS Λ_{mon}-resolution of $(\Lambda_{mon})_0$, we know that the image of L^n in L^{n-1} is contained in $J/I_{mon}L^{n-1}$, for $n \geq 1$. In particular, if $L^n \neq (0)$, then $F(n) \geq F(n-1)+1$. Thus, $L^{m+1} = (0)$. Hence, $T_{m+1} = \emptyset$. Thus, we also have $P^{m+1} = Q^{m+1} = (0)$ and we see that the projective dimension of Λ_0, as a Λ-module, and the projective dimension of $(\Lambda_{mon})_0$, as a Λ_{mon}-module, is less than or equal to m.

It remains to show that, assuming that Λ_{mon} is F-determined, then the AGS Λ-resolution of Λ_0 is, in fact, minimal. By the argument given above, the minimality of the AGS Λ_{mon}-resolution of $(\Lambda_{mon})_0$, implies that, if $Q^n \neq (0)$, $F(n) \geq F(n-1)+1$. But then the image of Q^n in Q^{n-1} is contained in $(J/I)Q^{n-1}$ and the result follows. □

We remark that if $\cdots \to P^2 \to P^1 \to P^0 \to \Lambda_0 \to 0$ be a minimal graded Λ-resolution of Λ_0, then an easy induction argument shows that if $P^n \neq 0$, then the generators of P^n occur in degrees n or higher. Thus, we always assume that the functions F under consideration have the property that $F(n) \geq n$, for all $n \geq 0$. Moreover, we assume, without loss of generality, that each P^i can be generated degrees greater than or equal to i, since the only possible exceptions occur if $P^i = 0$. These remarks and conventions, allow us to redefine weakly F-determined to mean that, for each $n \geq 0$, P^n can be generated in degrees bounded below by n, and bounded above by $F(n)$. A generalization of these remarks can be found in the proposition 2.1 of [5]. This generalization was proved in great detail in the Appendix of [18].

We now apply Proposition 7 to the d-Koszul case.

Corollary 8. *Let $\Lambda = K\Gamma/I$, where I is a homogeneous ideal in $K\Gamma$ contained in J^2. If, for some admissible order, the associated monomial algebra Λ_{mon} is d-Koszul, then Λ is d-Koszul algebra, and the reduced Gröbner basis of I is concentrated in degree d. Moreover, the AGS Λ-resolution of Λ_0 is a minimal graded projective Λ-resolution.*

Surprisingly, the following partial converse of Proposition 7 is true.

Proposition 9. *Let $\Lambda = K\Gamma/I$ be as above. Suppose that $F : \mathbb{N} \to \mathbb{N}$ is a set function such that Λ is F-determined and assume that the AGS Λ-resolution of Λ_0 is minimal. Then Λ_{mon} is F-determined.*

Proof. The proof follows from Propositions 5 and 6. □

In general, homological properties of Λ do not translate to Λ_{mon}, but the above result and the next result are exceptions to this. We now state the converse to Corollary 8 and note that we do not assume that the AGS Λ-resolution of Λ_0 is minimal. A related result can be found in [3], Proposition 2.3.

Theorem 10. *Let $\Lambda = K\Gamma/I$ where I is a homogeneous ideal in $K\Gamma$. Assume that $>$ is an admissible order on $\mathcal{B}$ such that the reduced Gröbner basis of I is concentrated in degree d, where d is a positive integer greater than 1. Then Λ being d-Koszul implies that Λ_{mon} is d-Koszul.*

Proof. Let $\mathcal{G}$ be the reduced Gröbner basis of I with respect to $>$, T_2 be the set of tips of $\mathcal{G}$, and T_3 be the set of maximal overlaps of T_2 with respect to $\mathcal{G}$. Since the paths in T_2 are a set minimal generators of I_{mon} and every path in T_2 is of length d, by [12], we need only show that every element of T_3 is of length $d+1$.

Let $g_1, g_2 \in \mathcal{G}$ and $t_i = \text{tip}(g_i)$, for $i = 1, 2$. Assume that t_1 maximally overlaps t_2 with respect to $\mathcal{G}$ with $p = t_2s = rt_1$ the maximal overlap, where $r, s \in \mathcal{B}$. Assume that $\ell(p) > d+1$ and let $\ell(p) = d^*$. Note that $d^* < 2d$. We show that this assumption leads to a contradiction. Clearly, $rg_1 - g_2s$ is a homogeneous element of degree d^*. Since $\mathcal{G}$ is a Gröbner basis of I, there exist elements nonnegative integers, A and B, and elements, $\alpha_1, \ldots, \alpha_A, \beta_1, \ldots, \beta_B \in K$, $x_1, \ldots, x_A, y_1, \ldots, y_B, z_1, \ldots, z_B \in \mathcal{B}$, with $\ell(z_j) \geq 1$, for $j = 1, \ldots, B$, and $g'_1, \ldots, g'_A, g''_1, \ldots, g''_B \in \mathcal{G}$, such that

$$rg_1 - g_2s = \sum_{i=1}^{A} \alpha_i x_i g'_i + \sum_{j=1}^{B} \beta_j y_j g''_j z_j, \qquad (*)$$

where $\text{tip}(rg_1 - g_2s) < r\,\text{tip}(g_1)$. Since $\mathcal{G}$ is a reduced Gröbnerbasis, the right hand side of $(*)$ is unique.

Since the elements of $\mathcal{G}$ are all homogeneous of degree d, we may assume that $\ell(x_i) = d^* - d$, for $i = 1, \ldots, A$ and that $\ell(y_j) + \ell(z_j) = d^* - d$, for $j = 1, \ldots, B$. Now let

$$F = rg_1 - \sum_{i=1}^{A} \alpha_i x_i g'_i = g_2 s + \sum_{j=1}^{B} \beta_j y_j g''_j z_j.$$

We see that $\mathrm{tip}(F) = rt_1$, which is a path of length d^*.

Let $\{f_i^n\}_{i\in T_n}$ be given by the AGS Λ-resolution as in [15]. As shown in [14], we may obtain sets V_n with $V_n \subset T_n$ so that $\{f_i^n\}_{i\in V_n}$ correspond to a minimal graded projective Λ-resolution of Λ_0. As we stated in Section 2, we have, for each $n \geq 2$,

$$(\oplus_{i\in V_{n-1}} K\Gamma f_i^{n-1}) \cap (\oplus_{i\in V_{n-2}} If_i^{n-2}) = (\oplus_{i\in V_n} K\Gamma f_i^n) \oplus (\oplus_{i\in W_n} Rf_i{}^{\prime n}), \qquad (**)$$

where each $f_i{}^{\prime n}$ is in $\oplus_{i\in W_{n-1}} If_i^{n-1}$, and the W_n are given in [15] and [14]. Now, for $i = 0, 1$, $V_i = T_i$, since Γ_i is the set of f_j^i's in both the minimal and the AGS Λ-resolutions for Λ_0. Our assumption that the reduced Gröbner basis of I consists of homogeneous elements in one degree d, implies that $V_2 = T_2$, since, in this case, $\mathcal{G}$ is the set of f_j^2's for Λ_0 in both the minimal and the AGS Λ-resolutions for Λ_0. Applying $(**)$, we see that

$$(\oplus_{g\in\mathcal{G}} K\Gamma g) \cap (\oplus_{a\in\Gamma_1} Ia) = (\oplus_{i\in V_3} K\Gamma f_i^3) \oplus (\oplus_{i\in W_3} Rf_i^{3'}).$$

From the definition of F, we see that $F \in (\oplus_{g\in\mathcal{G}} K\Gamma g) \cap (\oplus_{a\in\Gamma_1} Ia)$. Thus,

$$F = \sum_{i\in V_3} h_i f_i^3 + \sum_{i\in W_3} h'_i f_i^{3'},$$

for some $h_i, h'_j \in K\Gamma$. But, since each $f_i^{3'} \in \oplus_{g\in\mathcal{G}} Ig$ and I is generated by $\mathcal{G}$, we conclude that the each $f_i^{3'}$ is homogeneous of length at least $2d$. Noting that $2d > d^*$, we conclude that each $h'_j = 0$ and

$$F = \sum_{i\in V_3} h_i f_i^3.$$

Our assumption that Λ is d-Koszul, implies that each f^3 is homogeneous of length $d+1$. Thus each h_i is homogeneous of length $d^* - d - 1 \geq 1$. But then $\mathrm{tip}(F) = \mathrm{tip}(h_i)\,\mathrm{tip}(f_i^3)$, for some $i \in V_3$. Now $\mathrm{tip}(F)$ is the maximal overlap of tips of $\mathcal{G}$ of length d^* and $\mathrm{tip}(f_i^3)$ is a maximal overlap of tips of $\mathcal{G}$ of length $d+1$. This is a contradiction since distinct maximal overlaps of elements of $\mathrm{tip}(\mathcal{G})$ cannot be subwords of one another. This completes the proof. □

A consequence of the results of this section is that, if the reduced Gröbner basis of an ideal I consists of elements, all homogeneous of one degree, then there is a finite check to determine whether or not $K\Gamma/I$ is d-Koszul.

The next proposition is related to the Proposition 2.3 in the work of Berger [3]. Also we note that in the language of the paper [4], to say that X is a Gröbner basis concentrated in degree d is equivalent to say that X is concentrated in degree d and it is confluent [4].

Proposition 11. *Let $\Lambda = K\Gamma/I$ and $>$ an admissible order such that the reduced Gröbner basis of I is concentrated in degree d. Then Λ is d-Koszul if and only if the set of maximal overlaps of elements of* $\mathrm{tip}(\mathcal{G})$ *with respect to $\mathcal{G}$ are all of length $d+1$.*

Proof. Let $\mathcal{G}$ be the reduced Gröbner basis of I with respect to $>$. If Λ is d-Koszul, the proof of the above theorem shows that every maximal overlap of tips of $\mathcal{G}$ is a path of length $d+1$. On the other hand, if the set of maximal overlaps of elements of $\mathrm{tip}(\mathcal{G})$ are all of length $d+1$, then since I_{mon} has $\mathrm{tip}(\mathcal{G})$ as its minimal generating set, by [12, Theorem 10.2], Λ_{mon} is d-Koszul. Then Λ is d-Koszul by Corollary 8. □

The following result summarizes the main ideas of this section.

Theorem 12. *Let $\Lambda = K\Gamma/I$ and $>$ an admissible order such that the reduced Gröbner basis of I is concentrated in degree d, with $d \geq 2$. Then the following statements are equivalent:*

(1) *Λ is a d-Koszul algebra.*
(2) *Λ_{mon} is a d-Koszul algebra.*
(3) *If $\cdots \to P^2 \to P^1 \to P^0 \to \Lambda_0 \to 0$ is a minimal graded Λ-projective resolution of Λ_0, then P^3 is generated in degree $d+1$.*
(4) *If $\cdots \to P^2 \to P^1 \to P^0 \to (\Lambda_{mon})_0 \to 0$ is a minimal graded Λ_{mon}-projective resolution of $(\Lambda_{mon})_0$, then P^3 is generated in degree $d+1$.*
(5) *If $\mathcal{G}$ is the reduced Gröbner basis of I with respect to $>$, the every maximal overlaps of two elements of* $\{\mathrm{tip}(\mathcal{G})\}$ *with respect to $\mathcal{G}$, is of length $d+1$.*

4. 2-*d*-determined monomial algebras are 2-*d*-Koszul

Let $\Lambda = K\Gamma/I$, where I is a homogeneous ideal. We keep the convention that $\delta\colon \mathbb{N} \to \mathbb{N}$ is defined by

$$\delta(n) = \begin{cases} \frac{n}{2}d & \text{if } n \text{ is even} \\ \frac{n-1}{2}d + 1 & \text{if } n \text{ is odd.} \end{cases}$$

We also let $\cdots \to P^2 \to P^1 \to P^0 \to \Lambda_0 \to 0$ be a minimal graded projective Λ-resolution of Λ_0. We say that Λ is 2-d-*determined* if Λ is weakly δ-determined; that is, for each $n \geq 0$, P^n can be generated by elements of degree at least n and not greater than $\delta(n)$.

In keeping with the philosophy that the use of the word 'Koszul' should imply that Ext-algebra is finitely generated, we say that a 2-d-determined algebra is a *2-d-Koszul algebra* if its Ext-algebra, $\bigoplus \mathrm{Ext}_{\Lambda}^{n\geq 0}(\Lambda_0, \Lambda_0)$ is finitely generated. We prove later in this section that if Λ is 2-d-determined monomial algebra, then Λ is a 2-d-Koszul algebra; in particular, we show that the Ext-algebra of Λ can be generated in degrees 0, 1, and 2.

For the remainder of this section, we restrict our attention to monomial algebras such that the minimal generating set of monomial relations occur in exactly two degrees, 2 and d, where d is an integer greater than 2. We fix the following notation for the remainder of this section. Let d be an integer greater than 2, K is a field, Γ is a quiver, I is a monomial ideal generated by paths of length 2 and paths of length d and $\Lambda = K\Gamma/I$. Since I is generated by monomials, there is a unique minimal set of generating paths, $\mathcal{G}$, such that $\mathcal{G}$ is the reduced Gröbner basis of I with respect to any admissible order on $\mathcal{B}$. Let $\mathcal{G}_2$ denote the set of paths of length 2 in I. Let $\mathcal{G}_d$ denote the set of paths of length d in $\mathcal{G}$. Note that $\mathcal{G}$, and hence $\mathcal{G}_2$ and $\mathcal{G}_d$ are independent of the choice of $>$ and that our assumption that I is a monomial ideal generated in degrees 2 and d implies that $\mathcal{G}_2 \cup \mathcal{G}_d = \mathcal{G}$.

Our first result gives necessary and sufficient conditions for the monomial algebra Λ to be 2-d-determined. Before giving the result, we recall Theorem 10.2 from [12], see also [3], Proposition 4.1.

Proposition 13. *Let $\Lambda = K\Gamma/I$, where I is a monomial ideal generated by a set, ρ, of paths of length d with $d \geq 2$. Then Λ is d-Koszul algebra if and only if, for each pair of paths $p, q \in \rho$, if $pr = sq$ with $1 \leq \ell(r) < d$ then every subpath of pr of length d is in ρ.*

Theorem 14. *Keeping the notations above, Λ is 2-d-determined if and only if the algebra $\Delta = K\Gamma/ < \mathcal{G}_d >$ is a d-Koszul algebra.*

Proof. For $n \geq 0$, let $\mathrm{AP}_{\Delta}(n)$ be the admissible sets for $\rho = \mathcal{G}_d$, defined in Section 2 and $\mathrm{AP}_{\Lambda}(n)$ be the admissible sets for $\rho = \mathcal{G}$.

Let $n \geq 0$ and assume that Δ is a d-Koszul monomial algebra. We need to show that if $a_n \in \mathrm{AP}_{\Lambda}(n)$, then $n \leq \ell(a_n) \leq \delta(n)$. By definition of the $\mathrm{AP}_{\Delta}(n)$'s, the inequalities hold for $n = 0, 1, 2$. Assume by induction, that $n \geq 3$ and the inequalities hold for $n-2$ and $n-1$. There are unique elements

$$a_{n-1} \in \mathrm{AP}_{\Lambda}(n-1), \qquad a_{n-2} \in \mathrm{AP}_{\Lambda}(n-2), \qquad r, s \in \mathcal{B},$$

such that $a_n = ra_{n-1}$ and $a_{n-1} = sa_{n-2}$. Furthermore, there is some $a_2 \in \mathrm{AP}_{\Lambda}(2)$ such that a_2 maximally overlaps rs with respect to $\mathcal{G}$ and $a_n = a_2 t a_{n-2}$, for some path t. If $\ell(a_2) = 2$, then $\ell(a_n) = \ell(a_{n-1}) + 1$ and the result follows from induction. If $\ell(a_2) = d$ and $\ell(s) = 1$, then again the result follows. Finally if $\ell(a_2) = d$ and $\ell(s) > 1$, then $\ell(s) \leq d-1$ and it follows that s is a prefix of an element a_2' of AP(2) of length d. Hence a_2 overlaps a_2'. By Proposition 13 and the maximality of the overlap with respect to $\mathcal{G}$, we have that $a_2 = rs$. Thus $\ell(a_n) = d + \ell(a_{n-2})$. Our assumption implies that $\ell(a_{n-2}) \leq \delta(n-2) = ((n-2)d/2) + 1$ if n is even and $\ell(a_{n-2}) \leq \delta(n-2) = ((n-3)d/2)$ if n is odd, and the result follows.

Now we assume that Λ is 2-d-determined. By Proposition 13, it suffices to show that if $a_2, a_2' \in \mathcal{G}_d$ and a_2 overlaps a_2', then every subpath of length d is in $\mathcal{G}_d$. Suppose that $a_2 r = s a_2'$ with $\ell(r) \geq 1$. We proceed by induction on the length of r. If $\ell(r) = 1$, then we are done. Suppose that $\ell(r) > 1$ and that $a_2 r = \alpha_{\ell(a_2 r)}\alpha_{\ell(a_2 r)-1}\cdots\alpha_2\alpha_1$, with the α_j's arrows. It suffices to show that $a_2^* = \alpha_{d+1}\alpha_d\cdots\alpha_2 \in \mathcal{G}_d$, since, if so, a_2 overlaps a_2^* with $a_2 r^* = s^* a_2^*$ and $\ell(r^*) = \ell(r) - 1$. Since $\ell(r) > 1$, $\ell(a_2 r) > \delta(3) = d+1$. Hence $a_2 r \notin \mathrm{AP}_{\Delta}(3)$ and we conclude that the overlap of a_2 with a_2' is not maximal with respect to $\mathcal{G}$. Thus there is $\hat{a}_2 \in \mathcal{G}$ that maximally overlaps a_2' with respect to $\mathcal{G}$. If $\ell(\hat{a}_2) = 2$, then $\hat{a}_2$ is a subpath of a_2, contradicting that $\mathcal{G}$ is a reduced Gröbner basis. Thus, $\hat{a}_2$ has length d and, since $\delta(3) = d+1$, we see that $\hat{a}_2 = \alpha_{d+1}\cdots\alpha_2$, as desired. $\square$

Since $\mathcal{G}_d$ being the Gröbner basis of d-Koszul algebra is equivalent to the length of every element in AP(3) (with respect to $\mathcal{G}_d$) having length exactly $d+1$, we have the following consequence of Theorem 14.

Corollary 15. *Let $\Lambda = K\Gamma/I$ be as in Theorem 14 and let $\cdots P^2 \to P^1 \to P^0 \to \Lambda_0 \to 0$ be a minimal graded projective Λ-resolution of Λ_0. The following statements are equivalent:*

(1) *The algebra Λ is 2-d-determined.*

(2) *The projective module P_3 in the minimal projective Λ-resolution of Λ_0 can be generated in degrees bounded above by $d+1$.*

Proof. Let $a_2, a_2' \in \mathcal{G} = \mathrm{AP}_{\Lambda}(3)$, where $\mathrm{AP}_{\Lambda}(n)$ are the admissible paths of order n with respect to $\mathcal{G}$. Suppose that a_2 maximally overlaps s_2' with respect to $\mathcal{G}$ and let $p \in \mathrm{AP}_{\Lambda}(3)$ be the overlap. If either $\ell(a_2) = 2$ or $\ell(a_2') = 2$, then $\ell(p) \leq d+1 = \delta(3)$. If both a_2 and a_2' are of length d, then the previous theorem and the properties of d-Koszul monomial algebras. $\square$

We now turn our attention to the Ext-algebra of a monomial 2-d-determined algebra. Recall that the *Ext-algebra of* Λ, which we denote by $E(\Lambda)$, is the algebra $\bigoplus_{n\geq 0} \mathrm{Ext}_{\Lambda}^{n}(\Lambda_0, \Lambda_0)$. We view $E(\Lambda)$ as a positively $\mathbb{Z}$-graded algebra, where $E(\Lambda)_n = \mathrm{Ext}_{\Lambda}^{n}(\Lambda_0, \Lambda_0)$. The next result shows that every 2-d-determined monomial algebra is a 2-d-Koszul algebra.

Theorem 16. *Let $\Lambda = K\Gamma/I$, where I is a monomial ideal generated by paths of 2 and d with $d \geq 3$. Then Λ is 2-d-determined if and only if Λ is 2-d-Koszul. In this case, $E(\Lambda)$ can be generated in degrees 0, 1, and 2.*

Proof. It suffices to show that if Λ is a 2-d-determined monomial algebra, then $E(\Lambda)$ can be generated in degrees 0, 1, and 2. Let $\mathcal{G}$ be the reduced Gröbner basis of I with respect to some admissible order. We let $\mathcal{G} = \mathcal{G}_2 \cup \mathcal{G}_d$, where $\mathcal{G}_2$ are the elements of $\mathcal{G}$ of degree 2 and $\mathcal{G}_d$ are the elements of $\mathcal{G}$ of degree d. Let $\mathrm{AP}_\Lambda(n)$ be the admissible paths with respect to $\mathcal{G}$. Suppose $n \geq 3$ and $a_n \in \mathrm{AP}_\Lambda(n)$. There are unique elements

$$a_{n-1} \in \mathrm{AP}_\Lambda(n-1), \qquad a_{n-2} \in \mathrm{AP}_\Lambda(n-2), \qquad r, s \in \mathcal{B},$$

such that $a_n = ra_{n-1}$ and $a_{n-1} = sa_{n-2}$. Furthermore, there is some $a_2 \in \mathrm{AP}_\Lambda(2)$ such that a_2 maximally overlaps rs with respect to $\mathcal{G}$ and $a_n = a_2ta_{n-2}$, for some path t. We also have that there is some $a_2^* \in \mathrm{AP}_\Lambda(2)$ such that $a_{n-1} = a_2^*p$, for some path p.

By [17], it suffices to show that either $a_n = a_2a_{n-2}$, or $a_n = a_1a_{n-1}$, for some $a_1 \in \mathrm{AP}_\Lambda(1) = \Gamma_1$. If $\ell(a_n) = \ell(a_{n-1}) + 1$, then $a_n = a_1a_{n-1}$, for some $a_1 \in \mathrm{AP}_\Lambda(1) = \Gamma_1$ and we are done. Suppose that $\ell(a_n) \geq \ell(a_{n-1}) + 2$. Then a_2 has length > 2 and we see that $\ell(a_2) = d$.

We must show that $\ell(t) = 0$; that is, we must show that t is a vertex. If $\ell(t) > 0$, then $\ell(a_{n-1}) > \ell(a_{n-2}) + 1$ and we conclude that $\ell(a_2^*) = d$. Hence, suppose that $\ell(a_2^*) = d$. We know that a_2 maximally overlaps rs with respect to $\mathcal{G}$. We have that $a_2^* = ss'$ for some path s'. Thus a_2 overlaps a_2^*. By our assumption that Λ is 2-d-determined, by Theorem 14, we see that $K\Gamma/\langle \mathcal{G}_d \rangle$ is d-Koszul. Hence, by Proposition 13, every path of length d in the overlap of a_2 with a_2^* is in $\mathcal{G}_d \subset \mathcal{G}$. In particular, this implies that t must be a vertex. This completes the proof. □

5. 2-*d*-determined algebras and Gröbner bases

Let $d > 2$, I be a homogeneous ideal in $K\Gamma$, and $\Lambda = K\Gamma/I$. In this section we study conditions on Λ that imply that Λ is 2-d-determined. At this time, we do not know if every 2-d-determined algebra is a 2-d-Koszul algebra. Throughout this section, we fix an admissible order $>$ on $\mathcal{B}$. Since, by definition, a 2-d-determined algebra is just a weakly δ-determined algebra, the next result is an immediate consequence of Proposition 7.

Proposition 17. *Keeping the above notations, if Λ_{mon} is 2-d-Koszul then Λ is 2-d-determined.*

We now prove another result that gives sufficient conditions for Λ to be 2-d-determined. Observe that we use here conditions for a monomial algebra to be d-Koszul, this was described in various places, see for instance Proposition 3.8 of [2], Proposition 4.1 of [3], of the characterization given in [12].

Theorem 18. *Let $\Lambda = K\Gamma/I$, where I is a homogeneous ideal in $K\Gamma$, and let $>$ be an admissible order on $\mathcal{B}$. Suppose that the reduced Gröbner basis $\mathcal{G}$ of I with respect to $>$ satisfies $\mathcal{G} = \mathcal{G}_2 \cup \mathcal{G}_d$ where $\mathcal{G}_2$ consists of homogeneous elements of degree 2 and $\mathcal{G}_d$ consists of homogeneous elements of degree d, where $d \geq 3$. Then Λ is 2-d-determined if $K\Gamma/\langle \mathrm{tip}(\mathcal{G}_d)\rangle$ is a d-Koszul algebra.*

Proof. We begin by showing that if $K\Gamma/\langle \mathrm{tip}(\mathcal{G}_d)\rangle$ is a d-Koszul algebra, then Λ_{mon} is 2-d-determined. By Theorem 14 and Proposition 17, this then shows that if $K\Gamma/\langle \mathrm{tip}(\mathcal{G}_d)\rangle$ is a d-Koszul algebra, then Λ is 2-d-determined. Assume $K\Gamma/\langle \mathrm{tip}(\mathcal{G}_d)\rangle$ is a d-Koszul algebra and, for $n \geq 0$, let $\mathrm{AP}_\Delta(n)$ be the admissible sets for $\rho = \mathrm{tip}(\mathcal{G}_d)$, defined in Section 2. Let $\mathrm{AP}_{\Lambda_{mon}}(n)$ be the admissible sets for $\rho = \mathrm{tip}(\mathcal{G})$.

We need to show that if $a_n \in \mathrm{AP}_{\Lambda_{mon}}(n)$, then $n \leq \ell(a_n) \leq \delta(n)$. By definition of the $\mathrm{AP}_{\Lambda_{mon}}(n)$'s, the inequalities hold for $n = 0, 1, 2$. Assume by induction, that $n \geq 3$ and the inequalities hold for $n-2$ and $n-1$. There are unique elements

$$a_{n-1} \in \mathrm{AP}_{\Lambda_{mon}}(n-1), \qquad a_{n-2} \in \mathrm{AP}_{\Lambda_{mon}}(n-2), \qquad r, s \in \mathcal{B},$$

such that $a_n = ra_{n-1}$ and $a_{n-1} = sa_{n-2}$. Furthermore, there is some $a_2 \in \mathrm{AP}_{\Lambda_{mon}}(2)$ such that a_2 maximally overlaps rs with respect to $\mathrm{tip}(\mathcal{G})$ and $a_n = a_2ta_{n-2}$, for some path t. If $\ell(a_2) = 2$, then $\ell(a_n) = \ell(a_{n-1}) + 1$ and the result follows from induction. If $\ell(a_2) = d$ and $\ell(s) = 1$, then again the result follows. Finally, if $\ell(a_2) = d$ and $\ell(s) > 1$, then $\ell(s) \leq d - 1$ and it follows that s is a prefix of an element a_2' of $\mathrm{AP}_{\Lambda_{mon}}(2)$ of length d. Hence a_2 overlaps a_2'. By Proposition 13 (applied to the d-Koszul algebra $K\Gamma/\langle \mathrm{tip}(\mathcal{G}_d)\rangle$) and by the maximality of the overlap with respect to $\mathrm{tip}(\mathcal{G})$, we have that $a_2 = rs$. Thus $\ell(a_n) = d + \ell(a_{n-2})$. Our assumption implies that $\ell(a_{n-2}) \leq \delta(n-2) = ((n-2)d/2) + 1$ if n is even, and $\ell(a_{n-2}) \leq \delta(n-2) = ((n-3)d/2)$ if n is odd, and we have shown that Λ_{mon} is 2-d-determined. □

Suppose, as in the theorem above, that $d \geq 3$ and $\mathcal{G}$ is the reduced Gröbner basis for a homogeneous ideal and that $\mathcal{G} = \mathcal{G}_2 \cup \mathcal{G}_d$, where $\mathcal{G}_2$ consists of quadratic elements and $\mathcal{G}_d$ consists of homogeneous elements of degree d. If $d > 3$, then $\mathcal{G}_2$ is the reduced Gröbner basis of the ideal it generates and hence, by [10], $K\Gamma/\langle \mathcal{G}_2 \rangle$ is a Koszul algebra. On the other hand, if $d = 3$, then $\mathcal{G}_2$ need not be the reduced Gröbnerbasis of the ideal it generates and it is not necessarily the case that $K\Gamma/\langle \mathcal{G}_2 \rangle$ is a Koszul algebra.

The next result is a partial converse to the above theorem.

Proposition 19. *Let $\Lambda = K\Gamma/I$, where I is a homogeneous ideal in $K\Gamma$, and let $>$ be an admissible order on $\mathcal{B}$. Suppose that the reduced Gröbner basis $\mathcal{G}$ of I with respect to $>$ satisfies $\mathcal{G} = \mathcal{G}_2 \cup \mathcal{G}_d$ where $\mathcal{G}_2$ consists of homogeneous elements of degree 2 and $\mathcal{G}_d$ consists of homogeneous elements of degree d, where $d \geq 3$. If Λ is 2-d-Koszul and the AGS Λ-resolution of Λ_0 is minimal, then $K\Gamma/\langle \mathrm{tip}(\mathcal{G}_d)\rangle$ is a d-Koszul algebra.*

Proof. We follow a line of reasoning similar to that found in the proof of Theorem 10. If $\{f_i^n\}_{i\in T_n}$ are given by the AGS Λ-resolution, which, we are assuming, is a minimal graded projective Λ-resolution of Λ_0. As we stated in Section 2, we have, for each $n \geq 2$,

$$(\oplus_{i\in T_{n-1}} K\Gamma f_i^{n-1}) \cap (\oplus_{i\in T_{n-2}} If_i^{n-2}) = (\oplus_{i\in T_n} K\Gamma f_i^n) \oplus (\oplus_{i\in W_n} Rf_i{}^{\text{‘}n}),$$

where each $f_i{}^{\text{‘}n}$ is in $\oplus_{i\in W_{n-1}} If_i^{n-1}$, and the W_n are given in [15,14]. We are assuming that the reduced Gröbner basis of I consists of homogeneous elements in two degrees, 2 and d. Our assumption that Λ is 2-d-determined, implies that every $f_3^i \in V_3$ is homogeneous of degree $\leq d+1$. In fact, from the construction of the f_3^i's, the homogeneous degrees of elements in T_3 are either 3 or $d+1$.

Suppose that Λ is 2-d-determined and the AGS Λ-resolution of Λ_0 is minimal. Let $\Delta = K\Gamma/\langle \text{tip}(\mathcal{G}_d)\rangle$. We wish to show that Δ is a d-Koszul algebra. For this, let $\text{AP}_\Delta(n)$ denote the admissible sequences for $\text{tip}(\mathcal{G}_d)$. It suffices to show that if $a_3 \in \text{AP}_\Delta(3)$, then $\ell(a_3) = d+1$ by Proposition 11. Let $a_3 \in \text{AP}_\Delta$ and suppose that $a_2, a_2' \in \text{AP}_\Delta(2)$ are such that a_3 is the maximal overlap of a_2' with a_2. Let $g_2, g_2' \in \mathcal{G}_d$ such that $\text{tip}(g_2) = a_2$ and $\text{tip}(g_2') = a_2'$. If p and q are paths such that $a_3 = a_2'p = qa_2$, then, since $\mathcal{G}$ is a reduced (homogeneous) Gröbnerbasis, there exist homogeneous elements $r_h, s_h, t_h \in K\Gamma$, for $h \in \mathcal{G}$ such that

$$qg_2 - g_2'p = \sum_{h\in\mathcal{G}} r_h h + \sum_{h\in\mathcal{G}} s_h h t_h,$$

with each term occurring has the same length and each nonzero s_h has length ≥ 1. Let $X = qg_2 - \sum_{h\in\mathcal{G}} = g_2'p + \sum_{h\in\mathcal{G}} s_h h t_h$. Then we see that $\text{tip}(X) = a_3$, X is one of the f_i^3's and hence of length $\leq d+1$. It follows that $\ell(a_3) \leq d+1$ and we are done. □

We can now put together our results in the following theorem.

Theorem 20. *Let $\Lambda = K\Gamma/I$, where I is a homogeneous ideal in $K\Gamma$, and let $>$ be an admissible order on $\mathcal{B}$. Suppose that the reduced Gröbner basis $\mathcal{G}$ of I with respect to $>$ satisfies $\mathcal{G} = \mathcal{G}_2 \cup \mathcal{G}_d$ where $\mathcal{G}_2$ consists of homogeneous elements of degree 2 and $\mathcal{G}_d$ consists of homogeneous elements of degree d, where $d \geq 3$. Then the following are true.*

(1) *If $K\Gamma/\langle\mathcal{G}_d\rangle$ is d-Koszul then Λ is a 2-d-determined algebra and $K\Gamma/\langle\text{tip}(\mathcal{G})\rangle$ is 2-d-Koszul.*
(2) *If $K\Gamma/\langle\text{tip}(\mathcal{G})\rangle$ is 2-d-Koszul, then Λ is 2-d-determined.*
(3) *If $\cdots \to P^3 \to P^2 \to P^1 \to P^0 \to (K\Gamma/\langle\mathcal{G}_d\rangle)_0$ is a minimal graded projective $K\Gamma/\langle\mathcal{G}_d\rangle$-resolution and P^3 can be generated in degree $\leq d+1$, then Λ is a 2-d-determined algebra.*

Assuming the AGS Λ-resolution of Λ_0 is minimal, the following statements are equivalent.

(4) *The algebra Λ is 2-d-determined.*
(5) *The algebra Λ_{mon} is 2-d-Koszul.*
(6) *The algebra $K\Gamma/\langle\text{tip}(\mathcal{G}_d)\rangle$ is d-Koszul.*

Proof. By Theorem 12 ((1) implies (2)), we see that $K\Gamma/\langle\mathcal{G}_d\rangle$ being d-Koszul implies that $K\Gamma/\langle\mathcal{G}_d\rangle_{mon}$ is d-Koszul. But $K\Gamma/\langle\mathcal{G}_d\rangle_{mon} = K\Gamma/\langle\text{tip}(\mathcal{G}_d)\rangle$ and part (1) follows from Theorem 18. Part (2) follows from Theorems 14 and 18. Part (3) follows from Corollary 15 and Theorem 18.

We have seen that (5) implies (6) and that (6) implies (4). That (4) implies (5), follows from Proposition 19. □

We end with the obvious questions:

Questions. Assume that $\Lambda = K\Gamma/I$, where I is a ideal generated by homogeneous elements of degrees 2 and d.

(i) If Λ is a 2-d-determined algebra, then, is the Ext-algebra $E(\Lambda) = \oplus_{n\geq 0}\text{Ext}_\Lambda^n(\Lambda_0, \Lambda_0)$ finitely generated?
(ii) If Λ is a 2-d-determined algebra and the Ext-algebra $E(\Lambda)$ finitely generated, is it generated in degrees 0, 1, and 2 (assuming that the global dimension of Λ is infinite).
(iii) If Λ is not of finite global dimension and $E(\Lambda)$ is generated in degrees 0, 1, and 2, then is Λ 2-d-determined?

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